

Implications of nonlinearity for spherically symmetric accretion

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ABSTRACT

Stationary solutions of spherically symmetric accretion processes have been subjected to a time-dependent radial perturbation, whose equation includes nonlinearity to any arbitrary order. Regardless of the order of nonlinearity, the equation of the perturbation bears a form that is remarkably similar to the metric equation of an analogue acoustic black hole. Casting the perturbation as a standing wave and maintaining nonlinearity in it up to the second order, bring out the time-dependence of the perturbation in the form of a Liénard system. A dynamical systems analysis of this Liénard system reveals a saddle point in real time, with the implication that instabilities will develop in the accreting system when the perturbation is extended into the nonlinear regime. The instability of initial subsonic states may also adversely affect the temporal evolution of the flow towards a final and stable transonic state.

Key words: accretion, accretion discs – black hole physics – hydrodynamics – instabilities – methods:analytical

1 INTRODUCTION

In the context of astrophysical fluid flows, the classical model of spherically symmetric accretion, proposed by Bondi (1952) sixty year ago, is in essence a mathematical problem of conservative and compressible hydrodynamics. This model has acquired the status of a paradigm in studies on accretion, and apart from the fact that it is amenable to exact mathematical analyses in many cases, the spherically symmetric model faithfully captures much of the physics of many astrophysical flows. So, notwithstanding its apparent simplicity, the spherically symmetric flow has been a subject of enduring interest to the researcher in astrophysical fluid dynamics, with multiple physical and mathematical variations on the original theme (Parker 1958; Salpeter 1964; Parker 1966; Axford & Newman 1967; Holzer & Axford 1970; Balazs 1972; Michel 1972; Mészáros 1975; Blumenthal & Mathews 1976; Mészáros & Silk 1977; Begelman 1978; Cowie et al. 1978; Stellingwerf & Buff 1978; Garlick 1979; McCray 1979; Brinkmann 1980; Moncrief 1980; Petterson et al. 1980; Vitello 1984; Bonazzola et al. 1987, 1992; Theuns & David 1992; Kazhdan & Murzina 1994; Ruffert 1994; Markovic 1995; Tsuribe et al. 1995; Titarchuk et al. 1996; Zampieri et al. 1996; Titarchuk et al. 1997; Kovalenko & Eremin 1998; Das 1999; Malec 1999; Toropin et al. 1999; Das 2000; Das & Sarkar 2001; Ray & Bhattacharjee 2002; Foglizzo 2002; Ray 2003; Babichev et al. 2004; Das 2004; Ray & Bhattacharjee 2005; Gaite 2006; Mandal et al. 2007; Roy 2007; Roy & Ray 2007; Naskar et al. 2007; Silich et al. 2008; Mach & Malec 2008; Roy 2011; Park & Ricotti 2011; Wong et al. 2011).

Accretion processes involve the flow dynamics of astrophysical matter under the external gravitational influence of an astrophysical object, like an ordinary star or a white dwarf or a neutron star or a black hole (Frank et al. 2002). Accretion flows are distinctly different from the self-gravity driven collapse of a fluid system, such as a star. The accreting astrophysical matter could be the interstellar matter, as modelled by its spherically symmetric infall onto an isolated accretor, or stellar matter, as seen in a binary system, where the tidal deformation of a star causes matter to flow out of it into the potential well of a compact companion (Frank et al. 2002). In all of these cases, the mathematical description of the fluid system involves a momentum balance equation (with gravity as an external force), the continuity equation and a polytropic equation of state (Frank et al. 2002).

Fluid flows, conservative or dissipative, fall under the general class of nonlinear dynamics. Set in full detail, the condition for momentum conservation in a fluid is a balance of dynamic effects, nonlinear effects and the effects of the pressure inherent in a continuum system (Landau & Lifshitz 1987). Prior to Bondi (1952), some studies of astrophysical flows had considered only the interplay between

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dynamics and nonlinearity, neglecting the effects of pressure with the argument that any heat generated would be radiated away rapidly, so that the temperature of the infalling gas (and related to it, the pressure as well) would remain negligibly low. The other extreme of ignoring dynamics and introducing pressure was taken up by Bondi (1952), in what became a stationary mathematical problem. Nonlinearity, however, was an abiding presence in either case.

While solving the stationary, spherically symmetric, compressible fluid flow was not difficult mathematically, interpreting the behaviour of the solutions from a physical perspective was. From the plethora of mathematical solutions in the stationary problem, the ones of physical relevance, in view of modelling astrophysical inflows, were identified to be locally subsonic very far away from the accretor. Within the class of inflows obeying this outer boundary condition, there is an infinitude of globally subsonic solutions, along which a fluid element may reach the accretor with a low subsonic velocity. However, for the same outer boundary condition, a single critical solution stands out in a class by itself, capable of allowing matter to reach the accretor with a high supersonic velocity, and crossing the sonic horizon along the way. This is the unique transonic solution — the classical Bondi (1952) accretion solution.

The exact fashion in which accreting matter reaches the accretor is related to the inner boundary condition of the inflow problem. In the case of the accretor being a black hole, the infall process must be transonic (Novikov & Thorne 1973; Shapiro & Teukolsky 1983). This is because a black hole has an event horizon instead of a physical surface, and thus precludes all possibility of a pressure build-up at small radii, that could otherwise have dominated over the free-fall conditions close to the accretor. The situation, however, is not so clearly understood if the accretor has a hard surface like a neutron star or a white dwarf. For such an accretor, it is supposed that the accumulated matter would build up pressure near the surface and cause the supersonic flow to be shocked down to subsonic levels, although for a neutron star in particular, all accreted matter is expected to be efficiently “vacuum cleaned” away, making it easier for the flow to remain supersonic (Pettersen et al. 1980). Evidently then, questions of setting the inner boundary condition and determining an inflow trajectory in relation to it, are not trivial ones to confront. Nevertheless, working with the stationary problem itself, Bondi (1952) had the insight that the transonic solution would be the one selected by a fluid element to reach the accretor from a distant outer boundary. The governing principles behind this choice were connected to the maximisation of the mass accretion rate, and the minimisation of the total energy configuration of the flow (Bondi 1952; Garlick 1979), although a definitive conclusion regarding the realisability of the transonic solution was left by Bondi (1952) to its stability.

The problem with the transonic solution in the stationary regime is that its realisability is notoriously vulnerable to even an infinitesimal deviation from the precisely needed boundary condition to generate the solution (Ray & Bhattacharjee 2002). This difficulty may be overcome by looking at the possibility of a temporal evolution of the accreting system towards the transonic state (Ray & Bhattacharjee 2002; Roy & Ray 2007). However, the nonlinear equations governing the temporal evolution of the flow do not lend themselves to ready mathematical analyses. Indeed, in the matter of incorporating both the dynamic and the pressure effects in the equations, the mathematical problem was very aptly described by Bondi (1952) as “insuperable”. So in the absence of any analytical formulation of the dynamics of the flow solutions, much of all time-dependent studies in spherically symmetric accretion is perturbative in character, based on linear stability analysis (Stellingwerf & Buff 1978; Garlick 1979; Pettersen et al. 1980; Ruffert 1994; Kovalenko & Eremin 1998; Foglizzo 2002; Ray 2003; Gaite 2006; Roy & Ray 2007), although in this respect, some non-perturbative studies have also been reported (Ray & Bhattacharjee 2002; Roy & Ray 2007). The range of the perturbative studies covers numerical and analytical methods, using both radial and non-radial perturbations, leading to varied conclusions about the behaviour of the perturbations in the spatial and temporal domains. The commonly accepted view to have emerged from all the linear stability analyses is that perturbations on the flow do not produce any linear mode with an amplitude that gets amplified in time (Gaite 2006), and that the perturbative method does not indicate the primacy of any particular class of solutions (Garlick 1979). This is as far as one could say, working in the linear regime. However, the general experience associated with any nonlinear system (and accreting systems are very much nonlinear) is that the understanding gained through linearised conditions can scarcely be imposed on circumstances dominated by nonlinearity. The work presented in this paper makes an attempt to bridge this gap.

In this work, a time-dependent, radial perturbation scheme implemented originally by Pettersen et al. (1980) has been adopted and all orders of nonlinearity have been retained in the resulting equation of perturbation. A most striking feature of the equation of the perturbation is that even on accommodating nonlinearity in full order, it conforms to the structure of the metric equation of a scalar field in Lorentzian geometry. This fluid analogue (an “acoustic black hole”), emulating many features of a general relativistic black hole, is a matter of continuing interest in fluid mechanics from diverse points of view (Moncrief 1980; Visser 1998; Schützhold & Unruh 2002; Barceló et al. 2005; Volovik 2005; Singha et al. 2005; Ray & Bhattacharjee 2007a,b; Roy & Ray 2007; Naskar et al. 2007; Das et al. 2007; Mach & Malec 2008).

The equation of the perturbation is then applied to study the stability of globally subsonic stationary solutions. Regarding the non-perturbative evolution of the accreting system, it is feasible to suggest that the initial condition of the evolution is a globally subsonic state, with gravity subsequently driving the system to a transonic state, sweeping through an infinitude of intermediate subsonic states. So, to ensure an unhindered temporal convergence to a stable transonic trajectory, the stability of the subsonic states is essential. To investigate this aspect at a relatively simple level, all orders of nonlinearity beyond the second order have been truncated in the equation of the perturbation. Following this, the spatial dependence of the perturbation has been integrated out with the help of well-defined boundary conditions on globally subsonic flows (Pettersen et al. 1980). After this, only the time-dependent part of the perturbation is extracted, and, very intriguingly, it acquires the mathematical appearance of a Liénard system (Strogatz 1994; Jordan & Smith 1999). Application of the common analytical tools of dynamical systems to study the equilibrium features of this Liénard system, shows the existence of a saddle point in real time, with the implication that the stationary background solutions will be unstable, if the perturbation is extended into the nonlinear regime.

So to summarise the import of this work, conservative momentum balance and continuity conditions, as appropriate for a stationary spherically symmetric flow, have been subjected to time-dependent radial perturbations. On including nonlinearity, an instability is seen

to develop in this otherwise simple hydrodynamic system. The entire mathematical treatment so described, and all its attendant physical conclusions, have been presented in what follows.

2 THE MATHEMATICAL CONDITIONS OF SPHERICALLY SYMMETRIC ACCRETION

The mathematical problem that was set up by Bondi (1952) himself and that is now taken up as a starting model in accretion-related texts (Chakrabarti 1990, 1996; Frank et al. 2002), involves two coupled fields, the local flow velocity, v , and the local density, ρ , of the compressible accreting fluid. These two coupled fields are governed by the continuity equation,

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (\rho v r^2) = 0, \quad (1)$$

and the inviscid Euler equation,

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} + \frac{1}{\rho} \frac{\partial P}{\partial r} + \Phi'(r) = 0, \quad (2)$$

tailored as they are, according to the requirements of spherical symmetry. In the latter equation, the local pressure, P , is expressed in terms of ρ , by invoking a general polytropic prescription, $P = k\rho^\gamma$, in which γ , the polytropic exponent, varies over the range (limited by isothermal and adiabatic conditions), $1 \leq \gamma \leq c_P/c_V$, with c_P and c_V being the two coefficients of specific heat capacity of a gas (Chandrasekhar 1939). The polytropic prescription is of a much more general scope than the simple conserved adiabatic case, and is suited well for the study of open systems like astrophysical flows. Now, making use of both P and ρ , it is also expedient to scale the flow velocity, v , in terms of a natural hydrodynamic scale of speed, c_s , which is the local speed of sound. This speed can be noted from $c_s^2 = \partial P / \partial \rho = \gamma k \rho^{\gamma-1}$.

The flow is driven by the gravity of a central accretor, whose potential is $\Phi(r)$. In equation (2) the driving force arising due to this potential is implied by its spatial derivative (represented by the prime). In the case of stellar accretion, the flow is driven by the Newtonian potential, $\Phi(r) = -GM/r$. On the other hand, quite often in studies of accretion onto a non-rotating black hole, it becomes convenient to dispense with the rigour of general relativity, and instead make use of a pseudo-Newtonian potential that mimics the general relativistic effects of Schwarzschild space-time geometry in a Newtonian construct of space and time (Paczynski & Wiita 1980; Nowak & Wagoner 1991; Artemova et al. 1996; Das & Sarkar 2001). The choice of a particular form of the pseudo-Newtonian potential, however, does not affect overmuch the general arguments regarding the stability of the flow.

With the functions, P and $\Phi(r)$, specified, equations (1) and (2) can give a complete description of the hydrodynamic flow in terms of the two fields, $v(r, t)$ and $\rho(r, t)$. From these dynamic variables, the steady solutions of the flow are obtained by making explicit time-dependence disappear, i.e. $\partial v / \partial t = \partial \rho / \partial t = 0$. The resulting differential equations, involving full spatial derivatives only, can then be easily integrated to get the stationary global solutions of the flow (Bondi 1952; Frank et al. 2002). A remarkable feature of these stationary solutions is that they remain invariant under the transformation $v \rightarrow -v$, i.e. the mathematical problem of inflows ($v < 0$) and outflows ($v > 0$) is identical in the steady state (Choudhuri 1999). This invariance has some adverse implications for critical flows in accretion processes. Critical solutions pass through saddle points in the stationary phase portrait of the flow (Ray & Bhattacharjee 2002; Roy & Ray 2007), but generating a stationary solution through a saddle point will be impossible by any physical means, because it calls for an infinite precision in the required outer boundary condition (Ray & Bhattacharjee 2002). Nevertheless, criticality is not a matter of doubt in accretion processes (Bondi 1952; Garlick 1979; Shapiro & Teukolsky 1983). The key to resolving this paradox lies in considering explicit time-dependence in the flow, because of which, as one may note from equations (1) and (2), the invariance under the transformation, $v \rightarrow -v$, breaks down. Obviously then, a choice of inflows ($v < 0$) or outflows ($v > 0$) has to be made at the very beginning (at $t = 0$, as it were), and solutions generated thereafter will be free of all the difficulties associated with the presence of a saddle point in the stationary flow.

On imposing various boundary conditions on the stationary integral solutions, multiple classes of flow result (Frank et al. 2002). Of these, the one that attracts attention in accretion studies obeys the boundary conditions, $v \rightarrow 0$ as $r \rightarrow \infty$ (the outer boundary condition) and $v > c_s$ for small values of r . It is quite obvious that this solution is transonic in nature, with its bulk flow velocity overcoming the local speed of sound at a particular point in space, r_c , the critical radius of the flow (Chakrabarti 1990; Frank et al. 2002; Ray & Bhattacharjee 2002). For a flow driven simply by the Newtonian potential, there is only one such critical radius. With the choice of a pseudo-Newtonian potential, multiple values of r_c could result, but practically speaking there would be only one physically relevant critical point, through which an integral solution could pass and attain the transonic state (Mandal et al. 2007).

It was argued by Bondi (1952) that among all the feasible stationary solutions by which a fluid element may reach the accretor, after having started under highly subsonic conditions on very large length scales, the actual trajectory chosen will be the one that is transonic in nature — the Bondi (1952) solution. This line of thinking was based on the criteria that with no restrictive inner boundary condition, the accretion rate will be as high as possible and the corresponding energy configuration of the flow shall be the lowest one (Garlick 1979). The transonic solution conforms to these requirements, taking into consideration only the stationary conditions. Under the approximation of a “pressureless” motion of a fluid in a gravitational field (Shu 1991), qualified support for transonicity also came later from a non-perturbative dynamic perspective (Ray & Bhattacharjee 2002; Roy & Ray 2007). No definitive conclusion about transonicity, however, can be drawn on the basis of a perturbative linear stability analysis (Garlick 1979).

Now, so far as generating the transonic flow is concerned, the non-perturbative dynamic evolution of global $v(r, t)$ and $\rho(r, t)$ profiles is very crucial indeed. Certainly, all the feasible stationary inflow solutions obey the outer boundary conditions that on large spatial scales,

$v(r) \rightarrow 0$ and $\rho(r) \rightarrow \rho_\infty$, where ρ_∞ is the constant “ambient” value of the density field very far away from the accretor (Frank et al. 2002). It is the way in which the two fields evolve close to the accretor that determines if the transonic state would be achieved or not. The dynamic process should be envisaged mathematically as one in which both the velocity and density fields, $v(r, t)$ and $\rho(r, t)$, are uniform initially for all values of r , in the absence of any driving force. Then with the introduction of a gravitational field (made effective at $t = 0$), the hydrodynamic fields, v and ρ , start evolving in time. If the temporal growth of v outpaces the temporal growth of ρ (to which c_s is connected) at small values of r , then the final stationary infall process will be transonic. Otherwise, the final stationary infall process will be globally subsonic, with $v(r) \rightarrow 0$ as $r \rightarrow 0$ (Petterson et al. 1980).

The non-perturbative evolution of the velocity and density fields in spherically symmetric accretion, however, requires working with a coupled set of nonlinear partial differential equations, as implied by equations (1) and (2). And where nonlinear equations are involved, one has to tread with caution, especially since no analytical solution of the dynamic problem exists in the case of spherically symmetric accretion.

3 NONLINEARITY IN THE PERTURBATIVE ANALYSIS

Equations (1) and (2) are easy to integrate in their stationary limits, and the resulting velocity and density fields, derived from these two equations, have only spatial profiles, $v \equiv v_0(r)$ and $\rho \equiv \rho_0(r)$. A standard practice in perturbative analysis (Petterson et al. 1980) is to apply small time-dependent, radial perturbations on the stationary profiles, $v_0(r)$ and $\rho_0(r)$, and then linearise the perturbed quantities. This, however, does not offer much insight into the time-dependent evolutionary aspects of the hydrodynamic flow. So the next logical step is to incorporate nonlinearity in the perturbative method. With the inclusion of nonlinearity in progressively higher orders, the perturbative analysis incrementally approaches the actual time-dependent evolution of the global solutions, after it has started with a given stationary profile at $t = 0$ (to make physical sense, this initial profile has to be very much subsonic at all spatial points).

The prescription for the perturbation is $v(r, t) = v_0(r) + v'(r, t)$ and $\rho(r, t) = \rho_0(r) + \rho'(r, t)$, in which the primed quantities indicate a perturbation about a stationary background. It is now necessary to define a new variable, $f(r, t) = \rho v r^2$, following a similar mathematical procedure employed by Petterson et al. (1980) and Theuns & David (1992). This variable emerges as a constant of the motion from the stationary limit of equation (1). This constant, f_0 , can be identified with the matter flow rate, within a geometrical factor (Frank et al. 2002), and in terms of v_0 and ρ_0 , it is given as $f_0 = \rho_0 v_0 r^2$. On applying the perturbation scheme for v and ρ , the perturbation in f , without losing anything of nonlinearity, is derived as

$$\frac{f'}{f_0} = \frac{\rho'}{\rho_0} + \frac{v'}{v_0} + \frac{\rho'}{\rho_0} \frac{v'}{v_0}. \quad (3)$$

The foregoing relation connects the perturbed quantities, v' , ρ' and f' , to one another. To get a relation between only ρ' and f' , one has to go back to equation (1), and apply the perturbation scheme on it. This will result in

$$\frac{\partial \rho'}{\partial t} = -\frac{1}{r^2} \frac{\partial f'}{\partial r}. \quad (4)$$

To obtain a similar relationship solely between v' and f' , one needs to combine the conditions given in equations (3) and (4), to get

$$\frac{\partial v'}{\partial t} = \frac{v}{f} \left(\frac{\partial f'}{\partial t} + v \frac{\partial f'}{\partial r} \right). \quad (5)$$

In equations (3), (4) and (5), all orders of nonlinearity have been maintained. Adhering to the same principle, applying the perturbation scheme in equation (2) and taking its second-order partial time derivative will yield

$$\frac{\partial^2 v'}{\partial t^2} + \frac{\partial}{\partial r} \left(v \frac{\partial v'}{\partial t} + \frac{c_s^2}{\rho} \frac{\partial \rho'}{\partial t} \right) = 0. \quad (6)$$

In deriving this expression, all the terms involved in the stationary flow have vanished due to taking a partial time derivative. This is slightly different from the practice of extracting the stationary part of equation (2) and making it disappear by setting its value as zero. Now making use of equations (4), (5) and the second partial time derivative of equation (5), a fully nonlinear equation of the perturbation is obtained from equation (6), in a symmetric form going as

$$\frac{\partial}{\partial t} \left(h^{tt} \frac{\partial f'}{\partial t} \right) + \frac{\partial}{\partial t} \left(h^{tr} \frac{\partial f'}{\partial r} \right) + \frac{\partial}{\partial r} \left(h^{rt} \frac{\partial f'}{\partial t} \right) + \frac{\partial}{\partial r} \left(h^{rr} \frac{\partial f'}{\partial r} \right) = 0, \quad (7)$$

in which,

$$h^{tt} = \frac{v}{f}, \quad h^{tr} = h^{rt} = \frac{v^2}{f}, \quad h^{rr} = \frac{v}{f} (v^2 - c_s^2). \quad (8)$$

Going by the symmetry of equation (7), it can be recast in a compact form as

$$\partial_\mu (h^{\mu\nu} \partial_\nu f') = 0, \quad (9)$$

with the Greek indices running from 0 to 1, under the equivalence that 0 stands for t and 1 stands for r . Equation (9), or equivalently, equation (7), is a nonlinear equation containing arbitrary orders of nonlinearity in the perturbative expansion. All of the nonlinearity is carried in

the metric elements, $h^{\mu\nu}$, involving the exact field variables, v , c_s and f , as opposed to containing only their stationary background counterparts (Visser 1998; Schützhold & Unruh 2002). This is going into the realm of nonlinearity, because v and c_s depend on f , while f is related to f' . If one were to have worked with a linearised equation only, then $h^{\mu\nu}$ could be read simply from the symmetric matrix (Roy & Ray 2007),

$$h^{\mu\nu} = \frac{v_0}{f_0} \begin{pmatrix} 1 & v_0 \\ v_0 & v_0^2 - c_{s0}^2 \end{pmatrix}, \quad (10)$$

in which $c_{s0}(r)$ is the stationary value of the local speed of sound. Now, in Lorentzian geometry the d'Alembertian for a scalar field in curved space is expressed in terms of the metric, $g_{\mu\nu}$, as

$$\Delta\varphi \equiv \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \varphi), \quad (11)$$

with $g^{\mu\nu}$ being the inverse of the matrix implied by $g_{\mu\nu}$ (Visser 1998; Barceló et al. 2005). Comparing equations (9) and (11) with each other, one could look for an equivalence between $h^{\mu\nu}$ and $\sqrt{-g} g^{\mu\nu}$. What can easily be appreciated from this comparison is that equation (9) gives an expression for f' that is of the type given by equation (11). In the linear order, the metrical part of equation (9), as equation (10) shows it, may then be extracted, and its inverse will incorporate the notion of the sonic horizon of an acoustic black hole, when $v_0^2 = c_{s0}^2$. This point of view has features that are similar to the metric of a wave equation obtained by setting the velocity of an irrotational, inviscid and barotropic fluid flow as the gradient of a scalar potential, and then by imposing a perturbation on this scalar potential (Visser 1998; Barceló et al. 2005). In contrast to this approach of exploiting the conservative nature of the flow to craft a scalar potential, the derivation of equation (9) makes use of the continuity condition. The latter method is more robust because the continuity condition is based on matter conservation, which is a firmer conservation principle than that of energy conservation, on which the conventional scalar-potential approach is founded.

Either way, all of this indicates that the physics of supersonic acoustic flows closely corresponds to many features of black hole physics. All infalling matter crosses the event horizon of a black hole maximally, i.e. at the greatest possible speed. By analogy the same thing may be said of matter crossing the sonic horizon in spherically symmetric inflows. Indeed, a long-standing conjecture about spherically symmetric accretion on to a point sink is that the transonic solution crosses the sonic horizon at the greatest possible rate (Bondi 1952; Garlick 1979). That this fact can be appreciated for the accretion problem through a perturbative result is remarkable, because conventional wisdom would have it that perturbative techniques are inadequate here (Garlick 1979).

However, all of this is valid only as far as the linear ordering goes. When nonlinearity is to be accounted for, then instead of equation (10), it will be equations (8) which will define the elements, $h^{\mu\nu}$, depending on the order of nonlinearity that one wishes to retain (in principle one could go up to any arbitrary order). The first serious consequence of including nonlinearity is to lose the argument in favour of the transonic condition (an inflow solution crossing the sonic horizon), because the description of $h^{\mu\nu}$, as stated in equation (10), will not suffice any longer. This view is in perfect conformity with a numerical study conducted by Mach & Malec (2008) for the case of spherically symmetric accretion, in which it was shown that if the perturbations were to become strong then the analogy between the “sonic horizon” and the event horizon of a black hole would not hold. Nevertheless, a most remarkable fact has emerged in consequence of including nonlinearity in the perturbative analysis. It is that regardless of the order of nonlinearity that one may desire to go up to, the symmetric form of the Lorentzian metric equation will remain unchanged, as shown very clearly by equation (9). For the laboratory fluid problem of the hydraulic jump, a similar type of symmetry was shown to exist, going up to the second order of nonlinearity (Ray & Bhattacharjee 2007b).

4 STANDING WAVES ON STEADY GLOBAL INFLOWS

All physically relevant inflow solutions obey the outer boundary condition, $v(r) \rightarrow 0$ as $r \rightarrow \infty$. In addition, if the solution is globally subsonic, then the inner boundary condition is $v(r) \rightarrow 0$ as $r \rightarrow 0$. From the point of view of a gravity-driven evolution of an inflow solution to a transonic state, the subsonic flows have great importance, because the initial state of an evolution, as well as the intermediate states in the march towards transonicity, should realistically be subsonic. So the stability of globally subsonic solutions must have a significant bearing on how a transonic solution will develop eventually. Imposing an Eulerian perturbation on subsonic inflows, their stability was studied by Petterson et al. (1980), and the amplitude of the perturbation in this case was seen to maintain a constant profile in time. In that respect one may say that the solutions do not exhibit any obvious instability. However, it is never prudent to extend this argument too far, especially when one considers nonlinearity in the perturbative effects, as it rightly ought to be done in a fluid flow problem.

Now equation (7) gives a nonlinear equation of the perturbation, accommodating nonlinearity up to any desired order. This equation can be applied to study the stability of stationary subsonic flows in a nonlinear regime. Following the mathematical procedure of Petterson et al. (1980), the perturbation is designed to behave like a standing wave about a globally subsonic stationary solution, obeying the boundary condition that the spatial part of the perturbation vanishes at two radial points in the spherical geometry — one at a great distance from the accretor (the outer boundary), and the other very close to it (the inner boundary).

The mathematical treatment involving nonlinearity is to be confined to the second order only (the lowest order of nonlinearity). Even simplified so, the entire procedure will still carry much of the complications associated with a nonlinear problem. The restriction of not going beyond the second order of nonlinearity implies that $h^{\mu\nu}$ in equations (8) will contain primed quantities in their first power only. Taken together with equation (7), this will preserve all terms which are nonlinear in the second order. So, carrying out the necessary expansion of

$v = v_0 + v'$, $\rho = \rho_0 + \rho'$ and $f = f_0 + f'$ in equations (8) up to the first order only, and defining a new set of metric elements, $q^{\mu\nu} = f_0 h^{\mu\nu}$, one obtains

$$\partial_\mu (q^{\mu\nu} \partial_\nu f') = 0, \quad (12)$$

in which μ and ν are to be read just as in equation (9). In the preceding expression, the elements, $q^{\mu\nu}$, carry all the three perturbed quantities, ρ' , v' and f' . The next process to perform is to substitute both ρ' and v' in terms of f' , since equation (12) is over f' only. To make this substitution possible, first one has to make use of equation (3) to represent v' in terms of ρ' and f' in all $q^{\mu\nu}$. While doing so, the product term of ρ' and v' in equation (3) is to be ignored, because including it will raise equation (12) to the third order of nonlinearity. Once v' has been eliminated in this manner, one has to write ρ' in terms of f' . This can be done by invoking equation (4), with the reasoning that if ρ' and f' are both separable functions of space and time, with the time part being oscillatory (all of which are standard mathematical prescriptions in perturbative analysis), then

$$\frac{\rho'}{\rho_0} = \sigma(r) \frac{f'}{f_0}, \quad (13)$$

with σ being a function of r only (which lends a crucial advantage in simplifying much of the calculations to follow). The exact functional form of $\sigma(r)$ will be determined by the way the spatial part of f' is set up. It was shown by Petterson et al. (1980) that $\sigma(r)$ would indeed be a real function, going as $\sigma(r) = v_0 (v_0 \pm c_{s0})^{-1}$, when the spatial part of f' was cast as a power series in the WKB approximation. In any case, it stands to reason that when both ρ' and v' are real fluctuations, σ should likewise be real.

Following all of these algebraic details, the elements, $q^{\mu\nu}$, in equation (12), can finally be expressed entirely in terms of f' as

$$q^{tt} = v_0 \left(1 + \epsilon \xi^{tt} \frac{f'}{f_0} \right), \quad q^{tr} = v_0^2 \left(1 + \epsilon \xi^{tr} \frac{f'}{f_0} \right), \quad q^{rt} = v_0^2 \left(1 + \epsilon \xi^{rt} \frac{f'}{f_0} \right), \quad q^{rr} = v_0 (v_0^2 - c_{s0}^2) + \epsilon v_0^3 \xi^{rr} \frac{f'}{f_0}, \quad (14)$$

in all of which, ϵ has been introduced as a nonlinear “switch” parameter to keep track of all the nonlinear terms. When $\epsilon = 0$, only linearity remains. In fact, in this limit one converges to the familiar linear result implied by equation (10). In the opposite extreme, when $\epsilon = 1$, in addition to the linear effects, the lowest order of nonlinearity (the second order) becomes activated in equation (12), and the linearised stationary conditions of a “sonic horizon” get disturbed due to the nonlinear ϵ -dependent terms. This very feature has been tested numerically by Mach & Malec (2008). Equations (14) also contain the factors, $\xi^{\mu\nu}$, all of which are to be read as

$$\xi^{tt} = -\sigma, \quad \xi^{tr} = \xi^{rt} = 1 - 2\sigma, \quad \xi^{rr} = 2 - \sigma \left[3 + (\gamma - 2) \frac{c_{s0}^2}{v_0^2} \right]. \quad (15)$$

Taking equations (12), (14) and (15) together, a nonlinear equation of the perturbation is obtained, completed up to the second order, without the loss of any relevant term.

To render equation (12), along with all $q^{\mu\nu}$ and $\xi^{\mu\nu}$, into a workable form, it will first have to be written explicitly, and then divided throughout by v_0 . While doing so, the symmetry afforded by $\xi^{tr} = \xi^{rt}$ is also to be exploited. The desirable form of the equation of the perturbation should be such that its leading term would be a second-order partial time derivative of f' , with unity as its coefficient. To arrive at this form, an intermediate step will involve a division by $1 + \epsilon \xi^{tt} (f'/f_0)$, which, binomially, is the equivalent of a multiplication by $1 - \epsilon \xi^{tt} (f'/f_0)$, with a truncation applied thereafter. This is dictated by the simple principle that to keep only the second-order nonlinear terms, it will suffice to retain just those terms which carry ϵ in its first power. The result of this entire exercise is

$$\begin{aligned} \frac{\partial^2 f'}{\partial t^2} + 2 \frac{\partial}{\partial r} \left(v_0 \frac{\partial f'}{\partial t} \right) + \frac{1}{v_0} \frac{\partial}{\partial r} \left[v_0 (v_0^2 - c_{s0}^2) \frac{\partial f'}{\partial r} \right] + \frac{\epsilon}{f_0} \left\{ \xi^{tt} \left(\frac{\partial f'}{\partial t} \right)^2 + \frac{\partial}{\partial r} \left(\xi^{rt} v_0 \frac{\partial f'^2}{\partial t} \right) - \frac{v_0}{2} \frac{\partial \xi^{rr}}{\partial r} \frac{\partial f'^2}{\partial t} \right. \\ \left. + \frac{1}{2v_0} \frac{\partial}{\partial r} \left(\xi^{rr} v_0^3 \frac{\partial f'^2}{\partial r} \right) - 2 \xi^{tt} f' \frac{\partial}{\partial r} \left(v_0 \frac{\partial f'}{\partial t} \right) - \frac{\xi^{tt} f'}{v_0} \frac{\partial}{\partial r} \left[v_0 (v_0^2 - c_{s0}^2) \frac{\partial f'}{\partial r} \right] \right\} = 0, \end{aligned} \quad (16)$$

in which, if one were to set $\epsilon = 0$, then what would remain would be the linear solution discussed in detail by Petterson et al. (1980) and Theuns & David (1992). To progress further, a solution of $f'(r, t)$, separable in space and time, is to be applied. This will bear the form, $f'(r, t) = R(r)\phi(t)$. Using this separable solution in equation (16), then multiplying the resulting expression throughout by $v_0 R$, and then performing some algebraic simplifications by partial integrations, will finally lead to

$$\begin{aligned} \ddot{\phi} v_0 R^2 + \dot{\phi} \frac{d}{dr} (v_0 R)^2 + \phi \left\{ \frac{d}{dr} \left[\frac{v_0}{2} (v_0^2 - c_{s0}^2) \frac{dR^2}{dr} \right] - v_0 (v_0^2 - c_{s0}^2) \left(\frac{dR}{dr} \right)^2 \right\} \\ + \frac{\epsilon}{f_0} \left[\dot{\phi}^2 \xi^{tt} v_0 R^3 + \dot{\phi} \phi \left[\frac{d}{dr} (\xi^{rt} v_0^2 R^3) + \xi^{rt} \frac{v_0^2}{3} \frac{dR^3}{dr} - \xi^{tt} R \frac{d}{dr} (v_0 R)^2 \right] \right. \\ \left. + \phi^2 \left\{ v_0 (v_0^2 - a_0^2) \frac{dR}{dr} \frac{d}{dr} (\xi^{tt} R^2) - \xi^{rr} v_0^3 R \left(\frac{dR}{dr} \right)^2 - \frac{d}{dr} \left[\xi^{tt} \frac{v_0}{3} (v_0^2 - c_{s0}^2) \frac{dR^3}{dr} \right] + \frac{d}{dr} \left(\xi^{rr} \frac{v_0^3}{3} \frac{dR^3}{dr} \right) \right\} \right] = 0, \end{aligned} \quad (17)$$

in which the overdots indicate full derivatives in time. Quite evidently, equation (17) is a second-order nonlinear differential equation in both

space and time. The way forward now is to integrate all spatial dependence out of equation (17), and then study the nonlinear features of the time-dependent part. The integration over the spatial part will necessitate invoking two boundary conditions, one at a small value of r (close to the accretor), and the other when $r \rightarrow \infty$ (very far from the accretor). At both of these boundary points, the perturbation will have a vanishing amplitude in time. It was reasoned by Petterson et al. (1980) that globally subsonic inflow solutions offer conditions for the fulfilment of the two required boundary conditions, and simultaneously maintain a continuity of the background solution in the interim region. The boundary conditions will ensure that all the “surface” terms of the integrals in equation (17) will vanish (which explains the tedious mathematical exercise to extract several such “surface” terms). So after carrying out the required integration on equation (17), over the entire region trapped between the two specified boundaries, all that will remain is the purely time-dependent part, having the form,

$$\ddot{\phi} + \epsilon (\mathcal{A}\phi + \mathcal{B}\dot{\phi}) \dot{\phi} + \mathcal{C}\phi + \epsilon \mathcal{D}\phi^2 = 0, \quad (18)$$

in which the constants, \mathcal{A} , \mathcal{B} , \mathcal{C} and \mathcal{D} are to be read as

$$\begin{aligned} \mathcal{A} &= \frac{1}{f_0} \left(\int v_0 R^2 dr \right)^{-1} \int \left[\xi^{rt} \frac{v_0^2}{3} \frac{dR^3}{dr} - \xi^{tt} R \frac{d}{dr} (v_0 R)^2 \right] dr, \\ \mathcal{B} &= \frac{1}{f_0} \left(\int v_0 R^2 dr \right)^{-1} \int \xi^{tt} v_0 R^3 dr, \\ \mathcal{C} &= - \left(\int v_0 R^2 dr \right)^{-1} \int v_0 (v_0^2 - c_{s0}^2) \left(\frac{dR}{dr} \right)^2 dr, \\ \mathcal{D} &= \frac{1}{f_0} \left(\int v_0 R^2 dr \right)^{-1} \int \left[v_0 (v_0^2 - c_{s0}^2) \frac{dR}{dr} \frac{d}{dr} (\xi^{tt} R^2) - \xi^{rr} v_0^3 R \left(\frac{dR}{dr} \right)^2 \right] dr, \end{aligned} \quad (19)$$

respectively. The form in which equation (18) has been abstracted is that of a general Liénard system (Strogatz 1994; Jordan & Smith 1999). All the terms of equation (18), which carry the parameter, ϵ , have arisen in consequence of nonlinearity. When one sets $\epsilon = 0$, one readily regains the linear results presented by Petterson et al. (1980). However, to go beyond linearity, and to appreciate the role of nonlinearity in the perturbation, one now has to understand the Liénard system that equation (18) has brought forth.

5 EQUILIBRIUM IN THE LIÉNARD SYSTEM

The mathematical form of a Liénard system is like a damped nonlinear oscillator equation, going as (Strogatz 1994; Jordan & Smith 1999)

$$\ddot{\phi} + \epsilon \mathcal{H}(\phi, \dot{\phi}) \dot{\phi} + \mathcal{V}'(\phi) = 0, \quad (20)$$

in which, \mathcal{H} is a nonlinear damping coefficient (the retention of the parameter, ϵ , alongside \mathcal{H} , attests to the nonlinearity), and \mathcal{V} is the “potential” of the system (with the prime on it indicating its derivative with respect to ϕ). In the present study,

$$\mathcal{H}(\phi, \dot{\phi}) = \mathcal{A}\phi + \mathcal{B}\dot{\phi}, \quad (21)$$

and

$$\mathcal{V}(\phi) = \mathcal{C} \frac{\phi^2}{2} + \epsilon \mathcal{D} \frac{\phi^3}{3}, \quad (22)$$

with the constant coefficients, \mathcal{A} , \mathcal{B} , \mathcal{C} and \mathcal{D} having to be read from equations (19).

To investigate the properties of the equilibrium points resulting from equation (20), it will be necessary to decompose this second-order differential equation into a coupled first-order dynamical system. To that end, on introducing a new variable, ψ , equation (20) can be recast as (Jordan & Smith 1999)

$$\begin{aligned} \dot{\phi} &= \psi \\ \dot{\psi} &= -\epsilon (\mathcal{A}\phi + \mathcal{B}\psi) \psi - (\mathcal{C}\phi + \epsilon \mathcal{D}\phi^2). \end{aligned} \quad (23)$$

Equilibrium conditions are established with $\dot{\phi} = \dot{\psi} = 0$. For the dynamical system implied by equations (23), this will immediately lead to two equilibrium points on the ϕ – ψ phase plane. Labelling the equilibrium points with a \star superscript, one can easily see that $(\phi^\star, \psi^\star) = (0, 0)$ in one case, whereas in the other case, $(\phi^\star, \psi^\star) = (-\mathcal{C}/(\epsilon \mathcal{D}), 0)$. In effect, both the equilibrium points lie on the line, $\psi = 0$, and correspond to the turning points of $\mathcal{V}(\phi)$. Higher orders of nonlinearity will simply have the effect of proliferating equilibrium points on the line, $\psi = 0$. For the present case of second-order nonlinearity, one of the equilibrium points is located at the origin of the ϕ – ψ phase plane, while the location of the other will depend both on the sign and the magnitude of \mathcal{C}/\mathcal{D} .

Having identified the position of the two equilibriums points, the next task would be to understand their stability. To do so, both equilibrium points are to be subjected to small perturbations, following which a linear stability analysis will have to be carried out. The perturbation scheme on both ϕ and ψ is $\phi = \phi^\star + \delta\phi$ and $\psi = \psi^\star + \delta\psi$. Applying this scheme on equation (23), and then linearising in $\delta\phi$

and $\delta\psi$, will lead to the coupled linear dynamical system,

$$\begin{aligned}\frac{d}{dt}(\delta\phi) &= \delta\psi \\ \frac{d}{dt}(\delta\psi) &= -\mathcal{V}''(\phi^*)\delta\phi - \epsilon\mathcal{H}(\phi^*, \psi^*)\delta\psi,\end{aligned}\tag{24}$$

in which $\mathcal{V}''(\phi^*) = \mathcal{C} + 2\epsilon\mathcal{D}\phi^*$. Using solutions of the type, $\delta\phi \sim \exp(\omega t)$ and $\delta\psi \sim \exp(\omega t)$, in equations (24), the eigenvalues of the Jacobian matrix of the dynamical system follow as

$$\omega = -\epsilon\frac{\mathcal{H}}{2} \pm \sqrt{\epsilon^2\frac{\mathcal{H}^2}{4} - \mathcal{V}''(\phi^*)},\tag{25}$$

with $\mathcal{H} \equiv \mathcal{H}(\phi^*, \psi^*)$ having to be evaluated at the equilibrium points.

Once the eigenvalues have been determined, it is now a simple task to classify the stability of an equilibrium point by putting its coordinates in equation (25). The equilibrium point at the origin has the coordinates, $(0, 0)$. Using these coordinates in equation (25), the two roots of the eigenvalues are obtained as $\omega = \pm i\sqrt{\mathcal{C}}$. If $\mathcal{C} > 0$, then the eigenvalues will be purely imaginary quantities, and consequently, the equilibrium point at the origin of the ϕ - ψ plane will be a centre-type point (Jordan & Smith 1999). And indeed, when the stationary spherically symmetric inflow solution, about which the perturbation is constrained to behave like a standing wave, is globally subsonic, then $\mathcal{C} > 0$, because in this situation, $v_0^2 < c_{s0}^2$ (Pettersen et al. 1980). Therefore, the centre-type equilibrium point at the origin of the phase plane indicates that the standing wave will be purely oscillatory in time, with no change in its amplitude. This very conclusion was drawn by Pettersen et al. (1980) in their linearised analysis of the standing wave, and it could be arrived at equally correctly by setting $\epsilon = 0$ (the linear condition) in equation (25).

The centre-type point at the origin of the phase plane has confirmed the results known already. It is the second equilibrium point that offers some novelties. This equilibrium point is entirely an outcome of taking nonlinearity to its lowest order (the second order) in the standing wave. The coordinates of this equilibrium point in the phase plane are $(-\mathcal{C}/(\epsilon\mathcal{D}), 0)$, and using these coordinates in equation (25), the eigenvalues become specified as

$$\omega = \frac{\mathcal{AC}}{2\mathcal{D}} \pm \sqrt{\left(\frac{\mathcal{AC}}{2\mathcal{D}}\right)^2 + \mathcal{C}}.\tag{26}$$

Noting as before, that $\mathcal{C} > 0$, and that \mathcal{A} , \mathcal{C} and \mathcal{D} are all real quantities, the inescapable conclusion is that the eigenvalues, ω , are real quantities, with opposite signs. In other words, the second equilibrium point is a saddle point (Jordan & Smith 1999), and as such its implications may be far-reaching when it comes to generating the transonic solution.

To understand this, the first thing to note is that if the magnitude of the temporal part of the perturbation exceeds a certain critical value, i.e. if $|\phi| > |\mathcal{C}/\mathcal{D}|$, then the perturbation will undergo a divergence in one of its modes. In other words, the stationary subsonic global background solution will become unstable under the influence of the perturbation. This is how it must happen in the vicinity of a saddle point, and higher orders of nonlinearity (starting with the third order) will not smother this effect (Strogatz 1994; Jordan & Smith 1999). The best that one may hope for is that the instability may grow in time till it reaches a saturation level imposed by a higher order of nonlinearity, a feature that has a precedence in the laboratory fluid problem of the hydraulic jump (Volovik 2006; Ray & Bhattacharjee 2007b).

While all of this gives the perturbative perspective, the implications of the saddle point for the non-perturbative evolutionary dynamics are also noteworthy. It is evident that there can be no transonic solution without gravity driving the infall process. So from a dynamic point of view, gravity starts the evolution towards the transonic state from an initial (and arguably nearly uniform) subsonic state, far away from the critical conditions for transonicity. If, however, the subsonic states are to encounter a saddle point in the real-time dynamics, then that should hold adverse implications for reaching a stable and stationary transonic end, which is the Bondi (1952) solution.

To ponder on a final point regarding the Liénard system, under linearised conditions, the perturbation on globally subsonic flows maintains a constant amplitude. Viewed in the phase portrait, this feature translates into closed phase trajectories around a centre-type point. Now, from dynamical systems theory, centre-type points are known to be “borderline” cases (Strogatz 1994; Jordan & Smith 1999). In such situations, the linearised treatment will show apparently stable behaviour but an instability may emerge immediately on accounting for nonlinearity (Strogatz 1994; Jordan & Smith 1999). This is exactly what has happened in the perturbative study carried out here.

6 CONCLUDING REMARKS

Going by the form of the Liénard system derived in this work, it is easy to see that the number of equilibrium points will depend on the order of nonlinearity that one may wish to retain in the equation of the perturbation. In practice, however, the analytical task becomes formidable with the inclusion of every higher order of nonlinearity. Going up to the second order, an instability in real time appears undeniable, but then one must realise that this conclusion has been made regarding a purely inviscid and conservative flow. Now, actual fluid flows have viscosity as another important physical factor to influence their dynamics. In fact, fluid flows are usually affected both by nonlinearity and viscosity, occasionally as competing effects, and apropos of this point, it is to be noted that for a linearised perturbation in spherically symmetric inflows, viscosity helps in decaying the amplitude of the standing waves on globally subsonic solutions (Ray 2003). So the instability that

has been seen to arise because of nonlinearity could very well be offset by accounting for viscosity in the flow. This is not to say though that viscosity will always act as a saviour to preserve stability, because in one of the proposed models of axisymmetric accretion, viscosity has been known to destabilise the flow (Bhattacharjee & Ray 2007; Bhattacharjee et al. 2009). The contrasting role of viscosity goes much beyond questions of stability. Looking at the respective geometries in spherically symmetric flows and axisymmetric flows, one notices that while viscosity tends to inhibit the infall process in the former (Ray 2003), it aids infall in the latter (Shakura & Sunyaev 1973; Pringle 1981; Frank et al. 2002). Apart from viscous dissipation, the stability of accretion processes can, moreover, be affected by radiative processes, turbulence (Mészáros 1975; Mészáros & Silk 1977; Ray & Bhattacharjee 2005) and magnetohydrodynamics (Balbus & Hawley 1998).

As a matter of regular practice, stability of fluids is also studied by constraining a perturbation to behave like a travelling wave (Pettersson et al. 1980; Cross 1986; Ray & Bhattacharjee 2007b). At times, one encounters the surprising situation of a fluid flow being stable under one type of perturbation, but unstable under the effect of another (Cross & Hohenberg 1993; Ray & Bhattacharjee 2007b). With nonlinearity lending an additional aspect, these effects merit a close examination in future studies.

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